中原大學 97 學年度博士班入學考試

97/06/04 08:30~10:00 物理學系

誠實是我們珍視的美德，
我們喜愛「拒絕作弊，堅守正直」的你！

科目：量子物理

□可使用計算機，惟僅限不具可程式及多重記憶者　■不可使用計算機

一、計算題（共 4 大題，每大題 25 分）：

1. The Hamiltonian of a one-dimensional simple harmonic oscillator is $H=\frac{p^2}{2m}+\frac{m\omega^2}{2}x^2$. It is convenient to define to two operators $a=\sqrt{\frac{m\omega}{2\hbar}}\left(x+i\frac{p}{m\omega}\right)$, $a^+=\sqrt{\frac{m\omega}{2\hbar}}\left(x-i\frac{p}{m\omega}\right)$ known as the annihilation operator and the creation operator.

   (a) Please show that $[a,a^+]=1$ and $H=\hbar\omega(a^+a+\frac{1}{2})$. (5%)

   (b) We define the number operator $N=a^+a$. It is obvious that $H=\hbar\omega(N+\frac{1}{2})$. The eigenstate of N is denoted as $|n\rangle$ such that $N|n\rangle=n|n\rangle$. Please show that $a|n\rangle=\sqrt{n}|n-1\rangle$, $a^+|n\rangle=\sqrt{n+1}|n+1\rangle$. (5%)

   (c) Please show that n must be a nonnegative integer. (5%)

   (d) $\langle n|(\Delta x)^2|n\rangle=\langle n|x^2|n\rangle-(\langle n|x|n\rangle)^2$, $\langle n|(\Delta p)^2|n\rangle=\langle n|p^2|n\rangle-(\langle n|p|n\rangle)^2$. Please show that $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle=(n+\frac{1}{2})^2\hbar^2$. (5%)

   (e) In the Heisenberg picture, the operators are functions of time. Please show that $x(t)=x(0)\cos\omega t+\frac{p(0)}{m\omega}\sin\omega t$, $p(t)=-m\omega x(0)\sin\omega t+p(0)\cos\omega t$. (5%)

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2. From Schrödinger equation: $i\hbar\dfrac{\partial\psi(x,t)}{\partial t}=-\dfrac{\hbar^2}{2m}\nabla^2\psi(x,t)+V(x)\psi(x,t)$.

(a) Please show that $\dfrac{\partial\rho}{\partial t}+\nabla\cdot\vec{j}=0$, here $\vec{j}(x,t)=\dfrac{\hbar}{m}\mathrm{Im}(\psi^*\nabla\psi)$ and $\rho=\left|\psi(x,t)\right|^2$. (6%)

(b) Let us write $\psi(x,t)$ as $\psi(x,t)=\sqrt{\rho(x,t)}\exp\left[\dfrac{iS(x,t)}{\hbar}\right]$. Please show that

$\vec{j}(x,t)=\dfrac{\rho}{m}\nabla S$. (6%)

(c) If we substitute ψ by $\sqrt{\rho}e^{iS/\hbar}$, please show that the classical limit ($\hbar\to 0$) of

Schrödinger equation is $\dfrac{1}{2m}\left|\nabla S\right|^2+V(x)+\dfrac{\partial S(x,t)}{\partial t}=0$, if $\hbar\left|\nabla^2 S\right|<<\left|\nabla S\right|^2$. (6%)

(d) Under partial wave expansion, the wave function at large distance becomes:

$$\psi(x)\xrightarrow{|\vec{x}|\to\infty}\frac{1}{(2\pi)^{3/2}}\sum_{\ell}(2\ell+1)\frac{P_\ell(\cos\theta)}{2ik}\left[S_\ell(k)\frac{e^{ikr}}{r}-\frac{e^{-i(kr-\ell\pi)}}{r}\right].$$ Please show that

$\left|S_\ell(k)\right|=1$ according to the probability conservation. (7%)

3. Pauli matrices are $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Please show that $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$, $[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$. (6%)

(b) The 2×2 matrix representation of the rotation operator $\wp(\hat{n}, \phi)$ is $\exp(\frac{-i\vec{\sigma}\cdot\hat{n}}{2}\phi)$. Please show that

$$\exp(\frac{-i\vec{\sigma}\cdot\hat{n}}{2}\phi) = \begin{pmatrix} \cos\frac{\phi}{2} - in_z \sin\frac{\phi}{2} & (-in_x - n_y)\sin\frac{\phi}{2} \\ (-in_x + n_y)\sin\frac{\phi}{2} & \cos\frac{\phi}{2} + in_z \sin\frac{\phi}{2} \end{pmatrix} \text{ and } \exp(\frac{-i\vec{\sigma}\cdot\hat{n}}{2}\phi)\Big|_{\phi=2\pi} = -1$$

for any $\hat{n}$. (6%)

(c) $\vec{s} = \frac{\hbar}{2}\vec{\sigma}$. Please obtain $|\hat{n},+\rangle$ which satisfies $\vec{s}\cdot\hat{n}|\hat{n},+\rangle = \frac{\hbar}{2}|\hat{n},+\rangle$. (6%)

(d) Please describe how to apply spin precession to demonstrate that a 4π rotation is needed for the ket to return the original ket with the same sign. (7%)

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4. The electron in the hydrogen subjected to a uniform electric field in the positive z-direction is described by the following Hamiltonian: $H = H_0 + V$. $H_0 = \dfrac{p^2}{2m} + V_C(r)$, $V = -e|\vec{E}|z$. $V_C(r)$ is the Coulomb potential. Electric field strength is weak and V can be treated as perturbation.

(a) Please show that the 1_{st} order energy shift vanishes for the ground state. (6%)

(b) The 2_{nd} order energy shift can be written as $\dfrac{-1}{2}\alpha|\vec{E}|^2$, here

$\alpha = -2e^2 \sum_{k\neq 0}^{\infty} \dfrac{|\langle k^{(0)}|z|100\rangle|^2}{E_0^{(0)} - E_k^{(0)}}$. Please prove that $\alpha < \dfrac{16}{3}a_0^3$. (Hint: $E_n^{(0)} = -\dfrac{1}{n^2}\cdot\dfrac{e^2}{2a_0}$, $\langle r^2\rangle = a_0^2$.) (6%)

(c) For the first excited state ($n = 2$), there are four degenerate states $|2s\rangle$, $|2p, m=0\rangle$, $|2p, m=1\rangle$, $|2p, m=-1\rangle$. Please show the only nonvanishing matrix elements are $\langle 2p, m=0|V|2s\rangle$ and $\langle 2s|V|2p, m=0\rangle$. (6%)

(d) Please show that the zeroth-order kets that diagonalize V are

$|+\rangle = \dfrac{1}{\sqrt{2}}(|2s, m=0\rangle + |2p, m=0\rangle)$, $|-\rangle = \dfrac{1}{\sqrt{2}}(|2s, m=0\rangle - |2p, m=0\rangle)$, $|2p, m=+1\rangle$, $|2p, m=-1\rangle$ and please draw their schematic energy-level diagram. (7%)